

MULTIPLE EQUILIBRIA OF NONHOMOGENEOUS MARKOV CHAINS AND SELF-VALIDATING WEB RANKINGS

MARIANNE AKIAN, STÉPHANE GAUBERT, AND LAURE NINOVE

ABSTRACT. PageRank is a ranking of the webpages used by the search engine Google to determine in which order to display webpages. It measures how often a given webpage would be visited by a random surfer on the webgraph. It seems realistic that this ranking may influence the reputation of the webpages, and therefore the behavior of websurfers. We propose a simple model taking in account the mutual influence between webranking and websurfing.

We study this model by considering the following iteration on the set of stochastic row vectors: $\mathbf{r}(s+1) = \mathbf{u}_T(\mathbf{r}(s))$, where for all vector \mathbf{x} , $\mathbf{u}_T(\mathbf{x})$ is the unique invariant measure of a primitive stochastic matrix $M_T(\mathbf{x})$ constructed from the adjacency matrix of the webgraph. The parameter $T > 0$ is fixed and measures the confidence of the surfer in the ranking. We call T -PageRank the limit, if it exists, of these iterates when s tends to infinity. We also consider the simple iteration defined by $\tilde{\mathbf{r}}(s+1) = \tilde{\mathbf{r}}(s)M_T(\tilde{\mathbf{r}}(s))$.

We prove that, when T is large enough, the fixed point of these iterations is unique and the convergence is global on the domain. But for small values of T , at least when the matrix C is positive, there are always several fixed points.

Our analysis uses results of nonlinear Perron–Frobenius theory, Hilbert projective metric and Birkhoff’s coefficient of ergodicity.

1. INTRODUCTION

Motivation. The PageRank algorithm [6] is at the heart of one of the most popular Web search engines. Its basic idea is to use the graph structure of the Web in order to assign to each webpage a score. Basically, the PageRank score attributed to the webpages measures *how often a given page would be visited by a random walker on the webgraph*. Formally, let $C = [C_{ij}]$ be the $n \times n$ adjacency matrix of the webgraph, so that $C_{ij} = 1$ if there is a hyperlink from page i to page j and $C_{ij} = 0$ otherwise. For simplicity, we first assume that C is irreducible, that is the webgraph is strongly connected. Imagine that, when visiting a page i , a websurfer chooses randomly the next webpage he/she will visit, among the pages referenced by page i , with the uniform distribution. The trajectory of such a websurfer is a

Date: September 3, 2007.

2000 Mathematics Subject Classification. 15A51, 47H07, 15A48, 15A18, 47H12, 60J10, 68P20.

Key words and phrases. PageRank, stochastic matrix, Markov chain, stationary vector, iterative process, projective metric, Perron–Frobenius theory.

Part of the results of this paper were announced in the Proceedings of the 2nd Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 06) [1].

The first two authors were partially supported by the joint RFBR-CNRS grant number 05-01-02807. Part of this work was done while the third author was visiting INRIA. This author was supported by a FNRS research fellowship and partially by research programs IAP VI/4 DYSCO (Belgian Federal Science Policy) and ARC Large Graphs and Networks (Communauté Française de Belgique). The scientific responsibility rests with its authors.

Markov chain with transition matrix $M = [M_{ij}]$, given by

$$M_{ij} = \frac{C_{ij}}{\sum_k C_{ik}} .$$

In its most basic version, the PageRank vector \mathbf{r} , the entries of which give the PageRank score of the webpages, is simply defined as the stationary distribution of this random walk on the webgraph. Thus \mathbf{r} is the invariant measure of the irreducible matrix M , that is the unique stochastic vector such that

$$\mathbf{r} = \mathbf{r}M .$$

However, the assumption that a websurfer makes uniform draws on the webgraph may seem unrealistic: a websurfer could have an a priori idea of the value of webpages, therefore favoring pages from reputed sites. The webrank may influence the reputation of the websites, and hence it may influence the behavior of the websurfers, which ultimately may influence the webrank. In this paper, we propose a simple model taking into account the mutual influence between webranking and websurfing.

The T -PageRank. We consider a sequence of stochastic vectors, representing successive webranks, $\mathbf{r}(0), \mathbf{r}(1), \dots$, which is defined inductively as follows. The current ranking $\mathbf{r}(s)$ induces a random walk on the webgraph. We assume that the websurfer moves from page i to page j with probability proportional to $C_{ij}e^{E(\mathbf{r}(s)_j)/T}$, where E is an increasing function, the *energy*, and $T > 0$ is a fixed positive parameter, the *temperature*. The websurfer's trajectory is therefore a Markov chain with transition matrix $M(\mathbf{r}(s))$, where $M(\mathbf{x})$ is defined for all vector \mathbf{x} by

$$M(\mathbf{x})_{ij} = \frac{C_{ij}e^{E(\mathbf{x}_j)/T}}{\sum_k C_{ik}e^{E(\mathbf{x}_k)/T}} .$$

The unique stationary distribution of this Markov chain, i.e. the invariant measure of the matrix $M(\mathbf{r}(s))$, is then used to update the webrank. Thus,

$$(1a) \quad \mathbf{r}(s+1) = \mathbf{u}(\mathbf{r}(s)) ,$$

where, for all vector \mathbf{x} , $\mathbf{u}(\mathbf{x})$ is the unique stochastic vector such that

$$(1b) \quad \mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x})M(\mathbf{x}) .$$

We call T -PageRank the limit of $\mathbf{r}(s)$ when s tends to infinity, if it exists, or a fixed point of the map \mathbf{u} .

Note that if $\mathbf{r}(0)$ is the uniform distribution, then $\mathbf{r}(1)$ is the classical PageRank. Note also that the temperature T measures the randomness of the process. If T is small, with overwhelming probability, the websurfer shall move from page i to one of the pages j referenced by page i of best rank, i.e. maximizing $\mathbf{r}(s)_j$, whereas if $T = \infty$, the websurfer shall draw the next page among the pages j referenced by the page i , with the uniform distribution, as in the standard webrank definition. For $T = \infty$, the T -PageRank coincides with the classical PageRank, because in this case $M(\mathbf{r}(s)) = M$.

We also consider the simple iteration defined by

$$(2) \quad \tilde{\mathbf{r}}(s+1) = \mathbf{f}(\tilde{\mathbf{r}}(s)) , \quad \text{with } \mathbf{f}(\mathbf{x}) = \mathbf{x}M(\mathbf{x}) ,$$

where $\tilde{\mathbf{r}}(0)$ is an arbitrary stochastic vector. From a computational point of view, this is similar to the standard power method.

Main results. Our first main result shows that, if the temperature T is sufficiently large, the T -PageRank exists, is unique and does not depend on the initial ranking. Moreover, at least when the matrix C is primitive, the generalized power algorithm (2) can be used to compute the T -PageRank.

Theorem 1.1. *If $T \geq n \text{Lip}(E)$, where $\text{Lip}(E)$ is the Lipschitz constant of the function E , then the map \mathbf{u} given by (1b) has a unique fixed point and the iterates (1a) converge to it for every initial ranking. Moreover, if C is primitive and if T is large enough, the iterates (2) converge to this unique fixed point for every initial ranking.*

On the other hand, for small values of T , several T -PageRanks exist, depending on the choice of the initial ranking. In some cases, the T -PageRank does nothing but validating the initial “belief” in the interest of pages given by the initial ranking. Consider for instance the graph given in Figure 1 with adjacency matrix $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and let $E(x) = x$ for all x and $T = \frac{1}{4}$. Let $\mathbf{r}(0) = \tilde{\mathbf{r}}(0) = (\frac{1}{3} \ \frac{1}{3} + \varepsilon \ \frac{1}{3} - \varepsilon)$ for an arbitrary small $\varepsilon > 0$. Then the iterates (1) and (2) converge to a T -PageRank close to (0.021 0.978 0.001), so the initial belief that the node 2 is more interesting than node 3 has strongly increased. Our second main result shows that

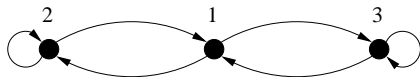


Figure 1: For this graph, self-validating effects appear for small temperatures.

the existence of multiple T -PageRanks is in fact a general feature, when T is small enough.

Theorem 1.2. *If C has at least two positive diagonal entries, then multiple T -PageRanks exist for T small enough.*

These two theorems follow respectively from Theorems 3.6 and 3.7, and from Theorem 3.10, which are stated in a more general framework.

We also study variants, bringing more realistic models of the behavior of the websurfer. This includes the presence of a “damping factor”, as in the standard definition of Google’s PageRank. We also consider the situation where the websurfer may take the webrank into account only when visiting a special page (the search engine’s webpage). Similar conclusions apply to such variants.

Method. In order to analyze the map (1b), we first use Tutte’s Matrix Tree Theorem [37] to express explicitly the invariant measure $\mathbf{u}(\mathbf{x})$ in term of the entries of $M(\mathbf{x})$. Then we study the convergence and the fixed points of (1) by using results of nonlinear Perron–Frobenius theory due to Nussbaum [31] and Krause [23]. To analyze the iteration (2), we use two different approaches, which give convergence results under distinct technical assumptions. Our first approach is to show that, if T is sufficiently large, the map \mathbf{f} from the simplex to itself is a contraction for some norm. Our second approach uses Hilbert’s projective metric and Birkhoff’s coefficient of ergodicity.

Related work. Several variants of the PageRank are considered in the literature in order to have a more realistic model of the behavior of the websurfers. For instance, some authors propose to introduce the browser’s back button in the model [11, 5, 36]. Other try to develop a topic-sensitive ranking by considering a surfer whose trajectory on the webgraph depends on his query or bookmarks [15, 20, 33]. Moreover, the standard PageRank vector is simply the dominant left-eigenvector of some stochastic matrix. It has therefore received attention from the linear algebra community. Current research concerns for instance efficient computation methods [18, 21, 25, 26, 35], or sensitivity analysis [22, 24, 25] of the PageRank.

Our use of transition probabilities proportional to $C_{ij}e^{E(\mathbf{r}(s)_j)/T}$, where E is an energy function and T the temperature, is reminiscent of simulated annealing algorithms. For a reference, see Catoni [9]. In the context of opinion formation, Holyst et al. [16] study a social impact model where the probability that an individual changes his opinion depends on a “social temperature” T , which measures the randomness of the process.

The iteration (1) can be studied in the settings of nonlinear Perron–Frobenius theory. Many works exist in the literature in order to generalize the classical Perron–Frobenius theorems to nonlinear maps on cone satisfying some hypotheses as, for instance, primitivity, positivity or homogeneity. See Nussbaum [31] and also [12, 29] for recent references.

The iteration (2) has been studied by several authors in an abstract setting. Artzrouni and Gavard [3] analyze their dynamics when $\mathbf{x}(s)$ behaves asymptotically like $\lambda^s \mathbf{x}_*$ for some $\lambda \neq 1$ and \mathbf{x}_* . For $\lambda = 1$, it can be useful to look at the stability of a linearization of the system near one of its fixed points [8]. When $M(\mathbf{x})$ is stochastic and satisfies certain monotonicity conditions, Conlisk [10] proves the convergence of the iterates (2) to a stable limit. Lorenz [27] proves their convergence for *column*-stochastic matrices satisfying classic properties of opinion dynamics models. In [28], he experimentally studies a reformulation of these models with stochastic matrices $M(\mathbf{x})$, where \mathbf{x} is an opinion distribution vector. Iterations like (2) could also be studied in the setting of nonhomogeneous products of matrices. In this case, iterations like $\mathbf{x}(s+1) = \mathbf{x}(s)M(s)$ are considered, where the matrices do not depend explicitly on \mathbf{x} . Two classical approaches to study their dynamics and convergence are the use of ergodicity coefficients [2, 14, 34, 39] or of the joint spectral radius [13, 14, 19]. However, the main results of this paper can not be deduced from these works.

Outline of the paper. We first introduce some preliminaries in Section 2. In Section 3, we prove the main results of this paper, about the existence, uniqueness or multiplicity of the T -PageRank, and the convergence of the iterates (1) and (2). Then, in Section 4, we introduce a refinement of our model, inspired by the damping factor of the classical PageRank algorithm. This variant has the advantage of allowing one to work with non strongly connected webgraphs. Moreover, it gives a more realistic model of the behavior of a standard web surfer. Section 5 is devoted to the estimation of the *critical temperature*, that is the temperature corresponding to the loss of the uniqueness of the T -PageRank, for some particular cases. We shall see that, even for very small or regular webgraphs, the T -PageRank can have a complex behavior. Finally, in Section 6, we experiment the T -PageRank algorithm on a large-scale example.

2. PRELIMINARIES

In this section, we introduce some notations and recall some classical concepts about nonnegative matrices and graphs, and about iterated maps on cones (for background, see [4, 17, 31, 34]).

We denote by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{> 0}$ the sets of nonnegative and positive numbers respectively. The *simplex* $\Sigma = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : \sum_i x_i = 1\}$ is the set of *stochastic row vectors*. Its relative interior is denoted by $\text{int}(\Sigma) = \Sigma \cap \mathbb{R}_{> 0}^n$, and its boundary by $\partial\Sigma = \Sigma \setminus \text{int}(\Sigma)$. We also write $\mathbf{x} \geq \mathbf{y}$ if the vector $\mathbf{x} - \mathbf{y} \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{x} > \mathbf{y}$ if the vector $\mathbf{x} - \mathbf{y} \in \mathbb{R}_{> 0}^n$. For some vector \mathbf{d} , let $D = \text{diag}(\mathbf{d})$ be the diagonal matrix such that $D_{ii} = d_i$ for all i and $D_{ij} = 0$ for all $i \neq j$. Finally, let $\mathbf{1}$ denote the column vector of all ones, and δ_{ij} denote the Kronecker delta.

Let us begin with nonnegative matrices and graphs [4, 17, 34]. Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be a nonnegative matrix. Its associated *directed graph* $G(A)$ is a graph with nodes $1, 2, \dots, n$, and a directed edge (i, j) if and only if $A_{ij} > 0$. This graph is *strongly connected*, that is there exists a directed path between each pair of nodes, if and only if A is *irreducible*. It is moreover *aperiodic* if and only if A is *primitive*. Let r be a node of $G(A)$. A directed subgraph R of $G(A)$ which contains no directed cycles and such that, for each node $i \neq r$, there is exactly one edge leaving i in R , is called a *spanning arborescence* rooted at r . The set of spanning arborescences of $G(A)$ rooted at r is denoted by $\mathcal{A}(r)$.

The transition probabilities of a *finite Markov chain*, or equivalently a *random walk* on a directed graph with weighted edges, can be represented by a *stochastic matrix*, that is a matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ such that $A\mathbf{1} = \mathbf{1}$. A stochastic vector $\mathbf{u} \in \Sigma$ is an *invariant measure* of A if $\mathbf{u}A = \mathbf{u}$. The Perron–Frobenius theory ensures that an irreducible stochastic matrix has a unique invariant measure, which is moreover positive. An expression of this invariant measure is given by Tutte’s Matrix Tree Theorem [37] (for a proof, see Section 9.6 of [7]).

Theorem 2.1 (Matrix Tree Theorem, Tutte). *Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be an irreducible stochastic matrix, and let \mathbf{u} be its invariant measure. Then $\mathbf{u} = \mathbf{v} / \sum_i v_i$, where for all index r*

$$(3) \quad v_r = \sum_{R \in \mathcal{A}(r)} \prod_{(i,j) \in R} A_{ij} \ .$$

We now present some useful concepts about iterated maps on cones [31, 34]. A function \mathbf{h} is *subhomogeneous* on a set U if $\lambda \mathbf{h}(\mathbf{x}) \leq \mathbf{h}(\lambda \mathbf{x})$ for all $\mathbf{x} \in U$ and every $0 \leq \lambda \leq 1$. It is *order-preserving* on U if $\mathbf{h}(\mathbf{x}) \leq \mathbf{h}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in U$ such that $\mathbf{x} \leq \mathbf{y}$. *Hilbert’s projective metric* d_H , is defined as

$$d_H : \mathbb{R}_{> 0}^n \times \mathbb{R}_{> 0}^n \rightarrow \mathbb{R}_{\geq 0} : (\mathbf{x}, \mathbf{y}) \mapsto \max_{i,j} \ln \frac{\mathbf{x}_i \mathbf{y}_j}{\mathbf{y}_i \mathbf{x}_j} \ .$$

This metric defines a distance on $\text{int}(\Sigma)$. An *induced projective metric* between two positive matrices can be defined as

$$d_H : \mathbb{R}_{> 0}^{n \times n} \times \mathbb{R}_{> 0}^{n \times n} \rightarrow \mathbb{R}_{\geq 0} : (A, B) \mapsto \sup_{\mathbf{x} \in \mathbb{R}_{> 0}^n} d_H(\mathbf{x}A, \mathbf{x}B) \ .$$

The *coefficient of ergodicity* τ_B , also known as Birkhoff's contraction coefficient, is defined for a nonnegative matrix A having no zero column as

$$\tau_B(A) = \sup_{\substack{\mathbf{x}, \mathbf{y} \in \mathbb{R}_{>0}^n \\ \mathbf{x} \neq \lambda \mathbf{y}}} \frac{d_H(\mathbf{x}A, \mathbf{y}A)}{d_H(\mathbf{x}, \mathbf{y})} .$$

This coefficient $\tau_B(A) \in [0, 1]$, and $\tau_B(A) < 1$ if and only if A is positive.

The following theorem deals with the fixed points and the convergence of iterated maps on a cone. This theorem is a simple formulation of the very general Theorems 2.5 and 2.7 (or more precisely of Corollaries 2.2 and 2.5) of Nussbaum in [31].

Theorem 2.2 (Nussbaum). *Let $\mathbf{h}: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ be a continuous, order-preserving map which is subhomogeneous on $\text{int}(\Sigma)$. Suppose moreover that if $\mathbf{x} \in \text{int}(\Sigma)$ is an eigenvector of \mathbf{h} , then \mathbf{h} is continuously differentiable on an open neighborhood of \mathbf{x} and the matrix $\mathbf{h}'(\mathbf{x})$ is nonnegative and irreducible. Then \mathbf{h} has at most one eigenvector in $\text{int}(\Sigma)$. As a consequence, the map $\mathbf{u}: \text{int}(\Sigma) \rightarrow \text{int}(\Sigma): \mathbf{x} \mapsto \mathbf{h}(\mathbf{x})/\sum_i \mathbf{h}(\mathbf{x})_i$ has at most one fixed point. Moreover, if \mathbf{h} has an eigenvector $\mathbf{x} \in \text{int}(\Sigma)$, and if $\mathbf{h}'(\mathbf{x})$ is primitive, then all the orbits of \mathbf{u} converge to its unique fixed point \mathbf{x} .*

We end with this preliminary section with a result of Krause [23], also dealing with the fixed point and the convergence of iterated maps on the positive orthant.

Theorem 2.3 (Krause). *Let $p: \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$ be a continuous map which is not identically 0 and such that $p(\lambda \mathbf{x}) = \lambda p(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}_{\geq 0}^n$, $\lambda \geq 0$; and $p(\mathbf{x}) \leq p(\mathbf{y})$ for all $0 \leq \mathbf{x} \leq \mathbf{y}$. Let $U = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^n: p(\mathbf{x}) = 1\}$. Let $\mathbf{h}: \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$ be a map such that there exist $\alpha, \beta > 0$ and $\mathbf{v} \in \mathbb{R}_{>0}^n$ such that $\alpha \mathbf{v} \leq \mathbf{h}(\mathbf{x}) \leq \beta \mathbf{v}$ for all $\mathbf{x} \in U$. Suppose also that $\lambda \mathbf{h}(\mathbf{x}) \leq \mathbf{h}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in U$ and $0 \leq \lambda \leq 1$ such that $\lambda \mathbf{x} \leq \mathbf{y}$; and that $\lambda \mathbf{h}(\mathbf{x}) < \mathbf{h}(\mathbf{y})$ if moreover $\lambda < 1$ and $\lambda \mathbf{x} \neq \mathbf{y}$. Then \mathbf{h} has a unique eigenvector in U . As a consequence, the map $\mathbf{u}: U \rightarrow \text{int}(U): \mathbf{x} \mapsto \mathbf{h}(\mathbf{x})/p(\mathbf{h}(\mathbf{x}))$ has a unique fixed point in U . Moreover, all the orbits of \mathbf{u} converge to this unique fixed point.*

3. EXISTENCE, UNIQUENESS, AND APPROXIMATION OF THE T -PAGERANK

3.1. Hypotheses. Let C be an $n \times n$ irreducible nonnegative matrix. For all temperature $T > 0$, and all $\mathbf{x} \in \Sigma$, let $M_T(\mathbf{x})$ be the irreducible stochastic matrix such that

$$M_T(\mathbf{x})_{ij} = \frac{C_{ij} g_T(\mathbf{x}_j)}{\sum_k C_{ik} g_T(\mathbf{x}_k)} ,$$

where $g_T: [0, 1] \rightarrow \mathbb{R}_{>0}$ is a continuously differentiable map with $g_T(0) = 1$. We suppose moreover that g_T is increasing with $g'_T: [0, 1] \rightarrow \mathbb{R}_{>0}$ and make the following assumptions on the asymptotic behavior of g_T

$$\begin{aligned} (A_0) \quad & \lim_{T \rightarrow 0} L_{\min}(g_T) = \infty , \\ (A_\infty) \quad & \lim_{T \rightarrow \infty} L_{\max}(g_T) = 0 , \end{aligned}$$

where $L_{\min}(g_T)$ and $L_{\max}(g_T)$ are defined as

$$L_{\min}(g_T) = \min_{x \in [0, 1]} \frac{g'_T(x)}{g_T(x)} \quad \text{and} \quad L_{\max}(g_T) = \max_{x \in [0, 1]} \frac{g'_T(x)}{g_T(x)} .$$

Example. These hypotheses are satisfied in particular for $g(x) = e^{E(x)/T}$, where the energy function E is continuously differentiable, with $E(0) = 0$ and $E'(x) > 0$ for all $x \in [0, 1]$. In this case, $L_{\min}(g_T) = \min_{x \in [0, 1]} E'(x)/T$ and $L_{\max}(g_T) = \max_{x \in [0, 1]} E'(x)/T$.

3.2. Preliminary results. The following elementary lemmas will be useful in the sequel.

Lemma 3.1. *For all $x, y \in [0, 1]$,*

$$L_{\min}(g_T)(x - y)^+ \leq \left(\ln \frac{g_T(x)}{g_T(y)} \right)^+ \leq L_{\max}(g_T)(x - y)^+ ,$$

where for all $x \in \mathbb{R}$, $x^+ = \max\{0, x\}$. Moreover, if $x, y \neq 0$,

$$\left(\ln \frac{g_T(x)}{g_T(y)} \right)^+ \leq L_{\max}(g_T) \left(\ln \frac{x}{y} \right)^+ .$$

Proof. Let $x, y \in [0, 1]$. Then

$$(\ln g_T(x) - \ln g_T(y))^+ \geq \min_{a \in [0, 1]} \frac{g'_T(a)}{g_T(a)} (x - y)^+ = L_{\min}(g_T)(x - y)^+ ,$$

$$(\ln g_T(x) - \ln g_T(y))^+ \leq \max_{a \in [0, 1]} \frac{g'_T(a)}{g_T(a)} (x - y)^+ = L_{\max}(g_T)(x - y)^+ .$$

Moreover, $(x - y)^+ \leq (\ln x - \ln y)^+$ if $x, y \in]0, 1]$. \square

Lemma 3.2. *Let $x, y \in [0, 1]$. If $x > y$, then $\lim_{T \rightarrow 0} g_T(x)/g_T(y) = \infty$.*

Proof. This result follows directly from Lemma 3.1 and Assumption (A_0) . \square

Lemma 3.3. *The map g_T tends to the constant function equal to 1, uniformly in $[0, 1]$, when T tends to infinity.*

Proof. For all $x \in [0, 1]$, by Lemma 3.1,

$$\ln g_T(x) = \ln \frac{g_T(x)}{g_T(0)} \leq L_{\max}(g_T)(x - 0) \leq L_{\max}(g_T) .$$

Therefore, by Assumption (A_∞) ,

$$\lim_{T \rightarrow \infty} \sup_{x \in [0, 1]} \ln g_T(x) \leq \lim_{T \rightarrow \infty} L_{\max}(g_T) = 0 . \quad \square$$

It follows from Lemma 3.3 that $\lim_{T \rightarrow \infty} M_T(\mathbf{x}) = \text{diag}(C\mathbf{1})^{-1}C$ uniformly for $\mathbf{x} \in \Sigma$.

3.3. Fixed points and convergence of \mathbf{u}_T . When C is irreducible, for all $\mathbf{x} \in \Sigma$, the matrix $M_T(\mathbf{x})$ is irreducible, and we can define the map

$$\mathbf{u}_T : \Sigma \rightarrow \Sigma : \mathbf{x} \mapsto \mathbf{u}_T(\mathbf{x}) ,$$

that sends \mathbf{x} to the unique invariant measure $\mathbf{u}_T(\mathbf{x})$ of $M_T(\mathbf{x})$. The Matrix Tree Theorem enables us to express explicitly $\mathbf{u}_T(\mathbf{x})$.

Lemma 3.4. *Assume that C is irreducible. Then, the invariant measure of $M_T(\mathbf{x})$ is given by*

$$\mathbf{u}_T(\mathbf{x}) = \frac{\mathbf{h}_T(\mathbf{x})}{\sum_k \mathbf{h}_T(\mathbf{x})_k} ,$$

where

$$(4) \quad \mathbf{h}_T(\mathbf{x})_r = \left(\sum_k C_{rk} g_T(\mathbf{x}_k) \right) \left(\sum_{R \in \mathcal{A}(r)} \prod_{(i,j) \in R} C_{ij} g_T(\mathbf{x}_j) \right) .$$

Proof. Apply Theorem 2.1 to $M_T(\mathbf{x})$, and take $\mathbf{h}_T(\mathbf{x}) = \mu \mathbf{v}$, where \mathbf{v} is given by (3) and $\mu = \prod_i \sum_k C_{ik} g_T(\mathbf{x}_k)$. \square

The existence of fixed points for \mathbf{u}_T is then proved using Brouwer's Fixed Point Theorem.

Proposition 3.5. *Assume that C is irreducible. The map \mathbf{u}_T has at least one fixed point in $\text{int}(\Sigma)$. Moreover, every fixed point of \mathbf{u}_T is in $\text{int}(\Sigma)$.*

Proof. By Lemma 3.4, the map $\mathbf{u}_T: \Sigma \rightarrow \Sigma$ is continuous, and therefore Brouwer's Fixed Point Theorem ensures the existence of at least one fixed point for \mathbf{u}_T . Moreover, since the invariant measure of an irreducible matrix is positive, and since $M_T(\mathbf{x})$ is irreducible, \mathbf{u}_T sends Σ to $\text{int}(\Sigma)$, and therefore every fixed point of \mathbf{u}_T is in $\text{int}(\Sigma)$. \square

The following result concerns the uniqueness of the fixed point and the convergence of the orbits of \mathbf{u}_T . With Assumption (A_∞) satisfied, it shows that the map \mathbf{u}_T has a unique fixed point and that all its orbits converge to this fixed point, for a sufficiently large temperature T . It can be proved using Nussbaum's Theorem 2.2, as we do it here, or, under the same hypotheses, using Krause's Theorem 2.3 (take $p(\mathbf{x}) = \sum_i \mathbf{x}_i$).

Theorem 3.6. *Assume that C is irreducible. If $nL_{\max}(g_T) \leq 1$, the map \mathbf{u}_T has a unique fixed point \mathbf{x}_T , which belongs to $\text{int}(\Sigma)$. Moreover all the orbits of \mathbf{u}_T converge to this fixed point.*

Proof. Since g_T is increasing, \mathbf{h}_T is an order-preserving map from $\mathbb{R}_{>0}^n$ to itself: $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{h}_T(\mathbf{x}) \leq \mathbf{h}_T(\mathbf{y})$. Now, let us show that \mathbf{h}_T is *subhomogeneous* on $\text{int}(\Sigma)$. Let $\mathbf{x} \in \text{int}(\Sigma)$ and $0 < \lambda \leq 1$. Any entry of $\mathbf{h}_T(\mathbf{x})$ is a sum of positively weighted terms like

$$\prod_k g_T(\mathbf{x}_k)^{\gamma_k} ,$$

with $\sum_k \gamma_k = n$. By Lemma 3.1, for each $k \in \{1, \dots, n\}$,

$$\ln \frac{g_T(\mathbf{x}_k)}{g_T(\lambda \mathbf{x}_k)} \leq L_{\max}(g_T) \ln \frac{1}{\lambda} .$$

Therefore, if $nL_{\max}(g_T) \leq 1$, then $\lambda^{1/n} g_T(\mathbf{x}_k) \leq g_T(\lambda \mathbf{x}_k)$, and it follows that $\lambda \mathbf{h}_T(\mathbf{x}) \leq \mathbf{h}_T(\lambda \mathbf{x})$. Since $0 \leq \mathbf{h}_T(0)$, this shows that \mathbf{h}_T is subhomogeneous on $\text{int}(\Sigma)$.

Finally, let \mathbf{x}_T be a fixed point of \mathbf{u}_T , by Proposition 3.5. The derivative $\mathbf{h}'_T(\mathbf{x})$ is a nonnegative continuous function of \mathbf{x} :

$$\begin{aligned} \frac{\partial \mathbf{h}_T(\mathbf{x})_r}{\partial \mathbf{x}_\ell} &= C_{r\ell} g'_T(\mathbf{x}_\ell) \left(\sum_{R \in \mathcal{A}(r)} \prod_{(i,j) \in R} C_{ij} g_T(\mathbf{x}_j) \right) \\ &+ \left(\sum_k C_{rk} g_T(\mathbf{x}_k) \right) \left(\sum_{R \in \mathcal{A}(r)} m_{\ell,R} \frac{g'_T(\mathbf{x}_\ell)}{g_T(\mathbf{x}_\ell)} \prod_{(i,j) \in R} C_{ij} g_T(\mathbf{x}_j) \right), \end{aligned}$$

where $m_{\ell,R} = |\{i: (i, \ell) \in R\}|$. Moreover, since g'_T takes positive values, $\partial \mathbf{h}_T(\mathbf{x})_r / \partial \mathbf{x}_\ell > 0$ as soon as $C_{r\ell} > 0$ or $m_{\ell,R} > 0$ for some $R \in \mathcal{A}(r)$. In particular, $m_{r,R} > 0$ for all $R \in \mathcal{A}(r)$. Since an irreducible matrix with positive diagonal is primitive, $\mathbf{h}'_T(\mathbf{x})$ is also primitive (see for instance Corollary 2.2.28 in [4]). Therefore, by Theorem 2.2, \mathbf{x}_T is the unique fixed point of \mathbf{u}_T , and all the orbits of \mathbf{u}_T converges to \mathbf{x}_T . \square

3.4. Fixed points and convergence of \mathbf{f}_T . We now consider the map

$$\mathbf{f}_T: \Sigma \rightarrow \Sigma: \mathbf{x} \mapsto \mathbf{x} M_T(\mathbf{x}) .$$

The following result shows that if T is sufficiently large, the fixed point of the map \mathbf{u}_T can be computed by iterating \mathbf{f}_T .

Theorem 3.7. *The fixed points of \mathbf{u}_T and \mathbf{f}_T are the same. If C is primitive, then, for T sufficiently large, all the orbits of \mathbf{f}_T converge to the fixed point \mathbf{x}_T of \mathbf{u}_T .*

Proof. Clearly, \mathbf{f}_T and \mathbf{u}_T have the same fixed points. Suppose now that C is primitive, and let us show then that, for T is sufficiently large, \mathbf{f}_T is a contraction for some particular norm. For every $\mathbf{x}, \mathbf{y} \in \Sigma$ and for any norm $\|\cdot\|$,

$$\|\mathbf{f}_T(\mathbf{x}) - \mathbf{f}_T(\mathbf{y})\| \leq \sup_{\mathbf{v} \in \Sigma} \|(\mathbf{x} - \mathbf{y}) \mathbf{f}'_T(\mathbf{v})\| .$$

The derivative of \mathbf{f}_T satisfies

$$\frac{\partial \mathbf{f}_T(\mathbf{v})_j}{\partial \mathbf{v}_\ell} = M_T(\mathbf{v})_{\ell j} + \sum_i (\delta_{\ell j} - M_T(\mathbf{v})_{ij}) \frac{C_{i\ell} \mathbf{v}_i g'_T(\mathbf{v}_\ell)}{\sum_k C_{ik} g_T(\mathbf{v}_k)} .$$

By Assumption (A_∞) and Lemma 3.3,

$$\lim_{T \rightarrow \infty} \mathbf{f}'_T(\mathbf{v}) = \lim_{T \rightarrow \infty} M_T(\mathbf{v}) = \text{diag}(C\mathbf{1})^{-1}C ,$$

uniformly for $\mathbf{v} \in \Sigma$. Let $A = \text{diag}(C\mathbf{1})^{-1}C$, and let $\mathcal{S} = \{\mathbf{z} \in \mathbb{R}^n: \sum_k \mathbf{z}_k = 0\}$ be the space of vectors orthogonal to the vector $\mathbf{1}$. The map $\mathbf{x} \mapsto \mathbf{x}A$ preserves the space \mathcal{S} , because $A\mathbf{1} = \mathbf{1}$. Let $A_{\mathcal{S}}$ denote the restriction of the map $\mathbf{x} \mapsto \mathbf{x}A$ to \mathcal{S} . Since the matrix A is primitive, its spectral radius, $\rho(A)$, is a simple eigenvalue and all the other eigenvalues of A have a strictly smaller modulus. Moreover, the space \mathcal{S} contains no eigenvector of A for the eigenvalue $\rho(A)$, because such an eigenvector \mathbf{u} must be a scalar multiple of the Perron eigenvector of A , which has positive entries, contradicting $\sum_k \mathbf{u}_k = 0$. We deduce that $\rho(A_{\mathcal{S}}) < \rho(A) = 1$. It follows that there exists a norm $\|\cdot\|$ such that $\|A_{\mathcal{S}}\| < 1$, where $\|\cdot\|$ is the matrix norm induced by $\|\cdot\|$ (see for instance Lemma 5.6.10 in [17]). Therefore, since $\mathbf{f}'_T(\mathbf{v})$ tends uniformly to A for $\mathbf{v} \in \Sigma$ when T tends to ∞ ,

$$\lim_{T \rightarrow \infty} \sup_{\mathbf{v} \in \Sigma} \|\mathbf{f}'_T(\mathbf{v})_{\mathcal{S}}\| = \|A_{\mathcal{S}}\| < 1 ,$$

where $\mathbf{f}'_T(\mathbf{v})_{\mathcal{S}}$ denotes the restriction of $\mathbf{f}'_T(\mathbf{v})$ on \mathcal{S} . It follows that, for all $\alpha \in]\|\mathbf{A}_{\mathcal{S}}\|, 1[$, there exists T_{α} such that for all $T > T_{\alpha}$,

$$\|\mathbf{f}_T(\mathbf{x}) - \mathbf{f}_T(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\| \sup_{\mathbf{v} \in \Sigma} \|\mathbf{f}'_T(\mathbf{v})_{\mathcal{S}}\| \leq \alpha \|\mathbf{x} - \mathbf{y}\| .$$

Hence, for such temperature T , by Banach's Fixed Point Theorem, \mathbf{f}_T has a unique fixed point and every orbit of \mathbf{f}_T converges to this fixed point. \square

Example. For the convergence of the orbits of \mathbf{f}_T , the primitivity of C cannot be dispensed with. Let for instance $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T > 0$. Then $M_T(\mathbf{x}) = C$ for all $\mathbf{x} \in \Sigma$, and $\mathbf{f}_T^k(\mathbf{x})$ oscillates when k tends to infinity, unless $\mathbf{x} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

For positive matrices $C \in \mathbb{R}_{>0}^{n \times n}$, let us now derive another convergence criterion, depending on Birkhoff's coefficient of ergodicity.

Lemma 3.8. *Assume that C is positive. Then, for any $\mathbf{x}, \mathbf{y} \in \text{int}(\Sigma)$,*

$$d_H(M_T(\mathbf{x}), M_T(\mathbf{y})) \leq 2L_{\max}(g_T) d_H(\mathbf{x}, \mathbf{y}) .$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \text{int}(\Sigma)$ be fixed. Let us define $\alpha = \max_{i,j} \frac{M_T(\mathbf{x})_{ij}}{M_T(\mathbf{y})_{ij}}$ and $\beta = \min_{i,j} \frac{M_T(\mathbf{x})_{ij}}{M_T(\mathbf{y})_{ij}}$. By definition,

$$\begin{aligned} d_H(M_T(\mathbf{x}), M_T(\mathbf{y})) &= \sup_{\mathbf{z} \in \mathbb{R}_{>0}^n} \max_{i,j} \ln \frac{(\mathbf{z} M_T(\mathbf{x}))_i (\mathbf{z} M_T(\mathbf{y}))_j}{(\mathbf{z} M_T(\mathbf{y}))_i (\mathbf{z} M_T(\mathbf{x}))_j} \\ &\leq \sup_{\mathbf{z} \in \mathbb{R}_{>0}^n} \max_{i,j} \ln \frac{(\alpha \mathbf{z} M_T(\mathbf{y}))_i (\mathbf{z} M_T(\mathbf{y}))_j}{(\mathbf{z} M_T(\mathbf{y}))_i (\beta \mathbf{z} M_T(\mathbf{y}))_j} = \ln \frac{\alpha}{\beta} . \end{aligned}$$

Moreover, by Lemma 3.1,

$$\begin{aligned} \ln \alpha &= \max_{i,j} \ln \left(\frac{C_{ij} g_T(\mathbf{x}_j)}{\sum_k C_{ik} g_T(\mathbf{x}_k)} \frac{\sum_k C_{ik} g_T(\mathbf{y}_k)}{C_{ij} g_T(\mathbf{y}_j)} \right) \\ &\leq \max_{j,k} \left(\ln \frac{g_T(\mathbf{x}_j)}{g_T(\mathbf{y}_j)} + \ln \frac{g_T(\mathbf{y}_k)}{g_T(\mathbf{x}_k)} \right) \\ &\leq L_{\max}(g_T) \left(\max_j \left(\ln \frac{\mathbf{x}_j}{\mathbf{y}_j} \right)^+ + \max_k \left(\ln \frac{\mathbf{y}_k}{\mathbf{x}_k} \right)^+ \right) \\ &= L_{\max}(g_T) \left(\left(\max_j \ln \frac{\mathbf{x}_j}{\mathbf{y}_j} \right)^+ + \left(\max_k \ln \frac{\mathbf{y}_k}{\mathbf{x}_k} \right)^+ \right) \\ &= L_{\max}(g_T) d_H(\mathbf{x}, \mathbf{y}) , \end{aligned}$$

since $\mathbf{x}, \mathbf{y} \in \text{int}(\Sigma)$ implies $\max_j \ln \frac{\mathbf{x}_j}{\mathbf{y}_j} \geq 0$ and $\max_k \ln \frac{\mathbf{y}_k}{\mathbf{x}_k} \geq 0$. We get similarly $-\ln \beta \leq L_{\max}(g_T) d_H(\mathbf{x}, \mathbf{y})$. \square

Proposition 3.9. *Assume that C is positive. If $2L_{\max}(g_T) < 1 - \tau_B(C)$, then \mathbf{f}_T has a unique fixed point $\mathbf{x}_T \in \text{int}(\Sigma)$ and all the orbits of \mathbf{f}_T converge to this fixed point.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \text{int}(\Sigma)$. By Lemma 3.8,

$$\begin{aligned} d_H(\mathbf{f}_T(\mathbf{x}), \mathbf{f}_T(\mathbf{y})) &\leq d_H(\mathbf{x} M_T(\mathbf{x}), \mathbf{y} M_T(\mathbf{x})) + d_H(\mathbf{y} M_T(\mathbf{x}), \mathbf{y} M_T(\mathbf{y})) \\ &\leq \tau_B(M_T(\mathbf{x})) d_H(\mathbf{x}, \mathbf{y}) + d_H(M_T(\mathbf{x}), M_T(\mathbf{y})) \\ &\leq (\tau_B(C) + 2L_{\max}(g_T)) d_H(\mathbf{x}, \mathbf{y}) . \end{aligned}$$

Therefore, if $2L_{\max}(g_T) < 1 - \tau_B(C)$, then \mathbf{f}_T is a contraction on $\text{int}(\Sigma)$ with respect to the distance d_H . By Banach's Fixed Point Theorem, \mathbf{f}_T has a unique fixed point $\mathbf{x}_T \in \text{int}(\Sigma)$ and all the orbits of \mathbf{f}_T converge to this fixed point. \square

3.5. Existence of multiple fixed points of \mathbf{u}_T and \mathbf{f}_T . Theorems 3.6 and 3.7 show that for a sufficiently large temperature T , the maps \mathbf{u}_T and \mathbf{f}_T have a unique fixed point. We can naturally wonder about the uniqueness of the fixed point of these maps for small T : we show that, at least when C is positive, multiple fixed points always exist.

Theorem 3.10. *Assume that C is irreducible and that the first column of C is positive. Then, for all $0 < \varepsilon < \frac{1}{2}$, there exists T_ε such that for $T \leq T_\varepsilon$, the map \mathbf{u}_T has a fixed point in $\Sigma_\varepsilon = \{\mathbf{x} \in \Sigma, \mathbf{x}_1 \geq 1 - \varepsilon\}$.*

Assume now that C is irreducible with $C_{11} > 0$ only, and that there exists $\varepsilon_n > 0$, independent of T , such that $g_T(\varepsilon_n)^{n-1} \leq g_T(1 - \varepsilon_n)$ for all $T > 0$. Then, for all $0 < \varepsilon < \varepsilon_n$, there exists T_ε such that for $T \leq T_\varepsilon$, the map \mathbf{u}_T has a fixed point in $\Sigma_\varepsilon = \{\mathbf{x} \in \Sigma, \mathbf{x}_1 \geq 1 - \varepsilon\}$.

Proof. Let k be the number of indices $i \neq 1$ such that $C_{i1} > 0$. Let $0 < \varepsilon < \frac{1}{2}$, and let $\mathbf{x} \in \Sigma_\varepsilon$. By the irreducibility of C , there exists a spanning arborescence R rooted at 1, containing all the k arcs $(i, 1)$ with $i \neq 1$ and $C_{i1} > 0$. Hence

$$\mathbf{h}_T(\mathbf{x})_1 \geq C_{11}g_T(\mathbf{x}_1) \prod_{(i,j) \in R} C_{ij}g_T(\mathbf{x}_j) \geq \alpha g_T(\mathbf{x}_1)^{k+1} \geq \alpha g_T(\mathbf{x}_1)^k g_T(1 - \varepsilon) ,$$

where $\alpha = C_{11} \prod_{(i,j) \in R} C_{ij} > 0$. Let $r \neq 1$. If $C_{r1} \neq 0$, then a spanning arborescence rooted at r can have at most $k - 1$ arcs $(i, 1)$ with $C_{i1} > 0$, whereas it can have at most k arcs $(i, 1)$ with $C_{i1} > 0$ in general. Hence, in all cases, $\mathbf{h}_T(\mathbf{x})_r$ is a sum of positively weighted terms like $\prod_\ell g_T(\mathbf{x}_\ell)^{\gamma_\ell}$ with $\sum_\ell \gamma_\ell = n$ and $\gamma_1 \leq k$. This implies that, for $r \neq 1$,

$$\mathbf{h}_T(\mathbf{x})_r \leq \beta g_T(\mathbf{x}_1)^k g_T(\varepsilon)^{n-k} ,$$

for some positive constant β . Therefore,

$$\mathbf{u}_T(\mathbf{x})_1 = \frac{1}{1 + \sum_{r \neq 1} \frac{\mathbf{h}_T(\mathbf{x})_r}{\mathbf{h}_T(\mathbf{x})_1}} \geq \frac{1}{1 + (n-1) \frac{\beta g_T(\varepsilon)^{n-k}}{\alpha g_T(1-\varepsilon)}} .$$

If the first column of C is positive, then $k = n - 1$. By Lemma 3.1,

$$\ln \frac{g_T(1 - \varepsilon)}{g_T(\varepsilon)} \geq L_{\min}(g_T) (1 - 2\varepsilon) .$$

Therefore, if $L_{\min}(g_T)(1 - 2\varepsilon) \geq \ln \frac{(n-1)\beta}{\alpha} \frac{1-\varepsilon}{\varepsilon}$, we get $\mathbf{u}_T(\mathbf{x})_1 \geq 1 - \varepsilon$. This shows that $\mathbf{u}_T(\Sigma_\varepsilon) \subset \Sigma_\varepsilon$. By Brouwer's Fixed Point Theorem, the continuous map \mathbf{u}_T has therefore at least one fixed point in Σ_ε .

Now, suppose we know only that $C_{11} > 0$, but there exists $\varepsilon_n > 0$ such that $g_T(\varepsilon_n)^{n-1} \leq g_T(1 - \varepsilon_n)$ for all $T > 0$. The map $\varphi_T: \varepsilon \mapsto (n-1) \ln g_T(\varepsilon) - \ln g_T(1 - \varepsilon)$ is increasing and its derivative satisfies $\varphi'_T(\varepsilon) \geq nL_{\min}(g_T)$. Let $0 < \varepsilon < \varepsilon_n$. We have $\varphi_T(\varepsilon_n) - \varphi_T(\varepsilon) \geq nL_{\min}(g_T)(\varepsilon_n - \varepsilon)$. Moreover, $k \geq 1$ by irreducibility of C and $\varphi_T(\varepsilon_n) \leq 0$, hence

$$\ln \frac{g_T(1 - \varepsilon)}{g_T(\varepsilon)^{n-k}} \geq \ln \frac{g_T(1 - \varepsilon)}{g_T(\varepsilon)^{n-1}} = -\varphi_T(\varepsilon) \geq nL_{\min}(g_T)(\varepsilon_n - \varepsilon) .$$

Therefore, if $nL_{\min}(g_T)(\varepsilon_n - \varepsilon) \geq \ln \frac{(n-1)\beta}{\alpha} \frac{1-\varepsilon}{\varepsilon}$, we get $\mathbf{u}_T(\mathbf{x})_1 \geq 1 - \varepsilon$. The result follows by the same argument as above. \square

Remark 3.1. When $g_T(x) = e^{E(x)/T}$ for some increasing energy E , then ε_n satisfies the condition $g_T(\varepsilon_n)^{n-1} \leq g_T(1 - \varepsilon_n)$ for all $T > 0$ if and only if $(n-1)E(\varepsilon_n) \leq E(1 - \varepsilon_n)$, which holds for some $0 < \varepsilon_n < 1$, since $E(0) = 0$ and $E(1) > 0$.

Corollary 3.11. If C is positive, then, for $T > 0$ sufficiently small, the map \mathbf{u}_T has several fixed points in Σ .

Example. If C is not positive, the existence of several fixed points for small T is not insured. Indeed, we shall see in Remark 5.1 that for $C = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $g_T(x) = e^{x/T}$, the fixed point of \mathbf{u}_T and \mathbf{f}_T is unique for each $T > 0$.

4. REFINEMENT OF THE MODEL

In the present section, we study a more general model, which includes a *damping factor* $0 < \gamma < 1$, as is the standard definition of Google's PageRank [6], in which the websurfer either jumps to the search engine with probability $1 - \gamma$ or moves to a neighbor page with probability γ . The presence of a damping factor yields a more realistic model of the websurfer walk. Moreover, it allows one to deal with reducible matrices, and it improves the convergence speed of iterative methods.

Let C be a $n \times n$ nonnegative matrix with no zero row and let $\mathbf{d} \in \mathbb{R}_{>0}^n$ be a personalization vector. For all temperature $0 < T < \infty$, let us define as previously $g_T: [0, 1] \rightarrow \mathbb{R}_{>0}$ as a continuously differentiable and increasing map, with $g_T(0) = 1$ and $g_T: [0, 1] \rightarrow \mathbb{R}_{>0}$. Suppose that Assumptions (A_0) and (A_∞) are satisfied. For a temperature $T = \infty$, let us also define $g_\infty(x) = 1$ for all $x \in [0, 1]$.

For any two temperatures $0 < T_1, T_2 \leq \infty$, and for all $\mathbf{x} \in \Sigma$, we can consider the positive transition matrix $M_{T_1, T_2, \gamma}(\mathbf{x})$ defined as

$$(5) \quad M_{T_1, T_2, \gamma}(\mathbf{x})_{ij} = \gamma \frac{C_{ij} g_{T_1}(\mathbf{x}_j)}{\sum_k C_{ik} g_{T_1}(\mathbf{x}_k)} + (1 - \gamma) \frac{\mathbf{d}_j g_{T_2}(\mathbf{x}_j)}{\sum_k \mathbf{d}_k g_{T_2}(\mathbf{x}_k)}.$$

Remark 4.1. For simplicity, we consider the same family of weight functions g_T for the first and the second term of $M_{T_1, T_2, \gamma}(\mathbf{x})$. Note however that the results of this section remain true if two families g_{T_1} and \tilde{g}_{T_2} are considered.

Remark 4.2. Suppose $T_1 = \infty$, $T_2 < \infty$ and $0 < \gamma < 1$, and let \mathbf{x} be the current ranking vector. Then $M_{T_1, T_2, \gamma}(\mathbf{x})$ is the transition matrix of the following random walk on the graph. At each step of his walk, either, with probability γ , the websurfer draws the next page uniformly among the pages referenced by his current page. Or, with probability $1 - \gamma$, he refers to the Web search engine, and therefore preferentially chooses for the next page a webpage with a good ranking.

The maps $\mathbf{u}_{T_1, T_2, \gamma}$ and $\mathbf{f}_{T_1, T_2, \gamma}$ are defined as previously: $\mathbf{u}_{T_1, T_2, \gamma}(\mathbf{x})$ is the unique invariant measure of $M_{T_1, T_2, \gamma}(\mathbf{x})$ and $\mathbf{f}_{T_1, T_2, \gamma}(\mathbf{x}) = \mathbf{x} M_{T_1, T_2, \gamma}(\mathbf{x})$.

Theorem 3.6 about the uniqueness of the fixed point of \mathbf{u}_T can be adapted in the following way.

Proposition 4.1. If $nL_{\max}(g_{T_1}) + (n-1)L_{\max}(g_{T_2}) \leq 1$, the map $\mathbf{u}_{T_1, T_2, \gamma}$ has a unique fixed point $\mathbf{x}_{T_1, T_2, \gamma}$ in Σ . Moreover, all the orbits of $\mathbf{u}_{T_1, T_2, \gamma}$ converge to the fixed point $\mathbf{x}_{T_1, T_2, \gamma}$.

Proof. If $T_1 = T_2 = \infty$, the result follows directly from Perron–Frobenius theory. Let us therefore suppose that $T_1 < \infty$ or $T_2 < \infty$. For every $\mathbf{x} \in \Sigma$, by Theorem 2.1, $\mathbf{u}_{T_1, T_2, \gamma} = \mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x}) / \sum_k \mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})_k$, where

$$\mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})_r = \left(\sum_k C_{rk} g_{T_1}(\mathbf{x}_k) \right) \left(\sum_{R \in \mathcal{A}(r)} \prod_{(i,j) \in R} W(\mathbf{x})_{ij} \right),$$

with $W(\mathbf{x})_{ij} = \sum_k (\gamma C_{ij} g_{T_1}(\mathbf{x}_j) \mathbf{d}_k g_{T_2}(\mathbf{x}_k) + (1 - \gamma) C_{ik} g_{T_1}(\mathbf{x}_k) \mathbf{d}_j g_{T_2}(\mathbf{x}_j))$.

Since g_{T_1} and g_{T_2} are nondecreasing, $\mathbf{h}_{T_1, T_2, \gamma}$ is an order-preserving map. Moreover, assume that $nL_{\max}(g_{T_1}) + (n - 1)L_{\max}(g_{T_2}) \leq 1$. Then, as in the proof of Theorem 3.6, $\mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})$ is shown to be subhomogeneous on $\text{int}(\Sigma)$.

Finally, the derivative $\mathbf{h}'_{T_1, T_2, \gamma}(\mathbf{x})$ is a nonnegative continuous function:

$$\begin{aligned} \frac{\partial \mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})_r}{\partial \mathbf{x}_\ell} &= C_{r\ell} g'_{T_1}(\mathbf{x}_\ell) \left(\sum_{R \in \mathcal{A}(r)} \prod_{(i,j) \in R} W(\mathbf{x})_{ij} \right) \\ &+ \left(\sum_k C_{rk} g_{T_1}(\mathbf{x}_k) \right) \left(\sum_{R \in \mathcal{A}(r)} \left(\prod_{(i,j) \in R} W(\mathbf{x})_{ij} \right) \left(\sum_{(i,j) \in R} \frac{\partial W(\mathbf{x})_{ij} / \partial \mathbf{x}_\ell}{W(\mathbf{x})_{ij}} \right) \right), \end{aligned}$$

where

$$\begin{aligned} \frac{\partial W(\mathbf{x})_{ij}}{\partial \mathbf{x}_\ell} &= \delta_{\ell j} \sum_k (\gamma C_{ik} g'_{T_1}(\mathbf{x}_k) \mathbf{d}_j g_{T_2}(\mathbf{x}_k) + (1 - \gamma) C_{ik} g_{T_1}(\mathbf{x}_k) \mathbf{d}_j g'_{T_2}(\mathbf{x}_j)) \\ &+ \gamma C_{ij} g_{T_1}(\mathbf{x}_j) \mathbf{d}_\ell g'_{T_2}(\mathbf{x}_\ell) + (1 - \gamma) C_{il} g'_{T_1}(\mathbf{x}_\ell) \mathbf{d}_j g_{T_2}(\mathbf{x}_j). \end{aligned}$$

Let us now prove that $\mathbf{h}'_{T_1, T_2, \gamma}(\mathbf{x})$ is a positive matrix for every \mathbf{x} . Suppose first that $T_2 < \infty$. Then $g'_{T_2}(\mathbf{x}_\ell) > 0$, and there exists a spanning arborescence $R \in \mathcal{A}(r)$ and a node i such that $(i, \ell) \in R$, since $G(M_{T_1, T_2, \gamma}(\mathbf{x}))$ is the complete graph. It follows that, for this R ,

$$\sum_{(i,j) \in R} \frac{\partial W(\mathbf{x})_{ij} / \partial \mathbf{x}_\ell}{W(\mathbf{x})_{ij}} \geq \frac{\partial W(\mathbf{x})_{i\ell} / \partial \mathbf{x}_\ell}{W(\mathbf{x})_{i\ell}} \geq \frac{\sum_k (1 - \gamma) C_{ik} g_{T_1}(\mathbf{x}_k) \mathbf{d}_\ell g'_{T_2}(\mathbf{x}_\ell)}{W(\mathbf{x})_{i\ell}} > 0,$$

and hence $\partial \mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})_r / \partial \mathbf{x}_\ell > 0$. Now, suppose that $T_2 = \infty$ and $T_1 < \infty$. Then we can suppose without loss of generality that C has no zero column (see Remark 4.3 below). Either $C_{r\ell} > 0$, and therefore $\partial \mathbf{h}_{T_1}(\mathbf{x})_r / \partial \mathbf{x}_\ell > 0$. Or there exists $i \neq r$ such that $C_{i\ell} > 0$, and for all $R \in \mathcal{A}(r)$, there exists j such that $(i, j) \in R$, that is

$$\frac{\partial W(\mathbf{x})_{ij}}{\partial \mathbf{x}_\ell} \geq (1 - \gamma) C_{i\ell} \mathbf{d}_j g'_{T_1}(\mathbf{x}_\ell) > 0,$$

and hence $\partial \mathbf{h}_{T_1, T_2, \gamma}(\mathbf{x})_r / \partial \mathbf{x}_\ell > 0$.

Since Brouwer's Fixed Point Theorem ensures the existence of at least one fixed point $\mathbf{x}_{T_1, T_2, \gamma} \in \text{int}(\Sigma)$ for the continuous map $\mathbf{u}_{T_1, T_2, \gamma}$ which sends Σ to $\text{int}(\Sigma)$, by Theorem 2.2, this fixed point $\mathbf{x}_{T_1, T_2, \gamma}$ is the unique fixed point of $\mathbf{u}_{T_1, T_2, \gamma}$, and all the orbits of $\mathbf{u}_{T_1, T_2, \gamma}$ converge to $\mathbf{x}_{T_1, T_2, \gamma}$. \square

Remark 4.3. If $T_2 = \infty$ and the matrix C has a zero column, the problem can be reduced to a problem of smaller dimension with a matrix with no zero column.

Indeed, suppose the n th column of C is zero. Then

$$\begin{aligned} (\mathbf{u}_{T_1, \infty, \gamma}(\mathbf{x})_1 \cdots \mathbf{u}_{T_1, \infty, \gamma}(\mathbf{x})_{n-1}) &= \frac{\sum_k \mathbf{d}_k - (1 - \gamma)\mathbf{d}_n}{\sum_k \mathbf{d}_k} \tilde{\mathbf{u}}_{T_1, \infty, \gamma}(\tilde{\mathbf{x}}) , \\ \mathbf{u}_{T_1, \infty, \gamma}(\mathbf{x})_n &= (1 - \gamma) \frac{\mathbf{d}_n}{\sum_k \mathbf{d}_k} , \end{aligned}$$

where $\tilde{\mathbf{x}} = (\mathbf{x}_1 \cdots \mathbf{x}_{n-1})$ and $\tilde{\mathbf{u}}_{T_1, \infty, \gamma}(\tilde{\mathbf{x}})$ is the invariant measure of a matrix $\tilde{M}_{T_1, \infty, \gamma}(\tilde{\mathbf{x}})$, with \tilde{C} the principal submatrix of C corresponding to the indices $1, \dots, n-1$, and $\tilde{\mathbf{d}}$ some positive vector of length $n-1$.

The following adaptations of Theorem 3.7 and Proposition 3.9 about the uniqueness of the fixed point of \mathbf{f}_T are quite direct.

Proposition 4.2. *The fixed points of $\mathbf{u}_{T_1, T_2, \gamma}$ and $\mathbf{f}_{T_1, T_2, \gamma}$ are the same. Moreover, for T_1 and T_2 sufficiently large, all the orbits of $\mathbf{f}_{T_1, T_2, \gamma}$ converge to the fixed point $\mathbf{x}_{T_1, T_2, \gamma}$ of $\mathbf{u}_{T_1, T_2, \gamma}$.*

Proposition 4.3. *Assume that C is positive. If $2(L_{\max}(g_{T_1}) + L_{\max}(g_{T_2})) < 1 - \tau_B(C)$, then $\mathbf{f}_{T_1, T_2, \gamma}$ has a unique fixed point $\mathbf{x}_{T_1, T_2, \gamma} \in \text{int}(\Sigma)$ and all the orbits of $\mathbf{f}_{T_1, T_2, \gamma}$ converge to the fixed point $\mathbf{x}_{T_1, T_2, \gamma}$.*

We next show that the map $\mathbf{f}_{T_1, T_2, \gamma}$ has multiple fixed points if either T_1 or T_2 is sufficiently small. Of course, this will require the damping factor to give enough weight to the terms corresponding to the small temperature in equation (5).

Proposition 4.4. *Assume that the first column of C is positive and that $\frac{1}{2} < \gamma < 1$. Then, there exists $T_0 > 0$ such that, for all $T_1 \leq T_0$ and for all $T_2 \in [0, \infty]$, the map $\mathbf{f}_{T_1, T_2, \gamma}$ has a fixed point in $\{\mathbf{x} \in \Sigma : \mathbf{x}_1 > \frac{1}{2}\}$.*

The conclusion of Proposition 4.4 is weaker than that of Theorem 3.10. The latter shows that for a sufficiently small temperature, we can find a fixed point of \mathbf{f}_T arbitrarily close to a vertex of the simplex, whereas the former shows that for a sufficiently small temperature, we can find a fixed point of $\mathbf{f}_{T_1, T_2, \gamma}$ in the region $\mathbf{x}_1 > 1/2$ of the simplex. In fact, such a fixed point may not approach a vertex of the simplex as the temperature tends to 0. This discrepancy is due to presence of the damping factor. The proof of Proposition 4.4 is a straightforward adaptation of the proof of Theorem 3.10.

Remark 4.4. *If the first column of C is not positive, the existence of a fixed point such that $\mathbf{x}_1 > \mathbf{x}_2, \dots, \mathbf{x}_n$ for small T_1 is not insured. Consider for instance $C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $g_T(x) = e^{x/T}$, $T_2 = \infty$, $\mathbf{d} = (1 \ 1)$ and $0 < \gamma < 1$. Then, for each $T_1 > 0$, any fixed point $\mathbf{x}_{T_1, \infty, \gamma}$ of $\mathbf{f}_{T_1, \infty, \gamma}$ belongs to $[0, \frac{1}{2}[\times]\frac{1}{2}, 1]$.*

We now show the existence of multiple fixed points for a sufficiently small temperature T_2 .

Proposition 4.5. *Let $0 < \varepsilon < \frac{1}{2}$ and let $\mu = \sum_{k \neq 1} \mathbf{d}_k / \mathbf{d}_1$. If $0 < \gamma < \varepsilon$, there exists $T_{\varepsilon, \gamma}$ such that for $T_2 \leq T_{\varepsilon, \gamma}$ and for all $T_1 \in [0, \infty]$, the map $\mathbf{f}_{\infty, T_2, \gamma}$ has a fixed point in $\Sigma_{\varepsilon} = \{\mathbf{x} \in \Sigma : \mathbf{x}_1 \geq 1 - \varepsilon\}$.*

Proof. Let $\mathbf{x} \in \Sigma_\varepsilon$. Since by Lemma 3.1, $\ln(g_{T_2}(1-\varepsilon)/g_{T_2}(\varepsilon)) \geq L_{\min}(g_{T_2})(1-2\varepsilon)$, we have

$$\begin{aligned} \mathbf{f}_{T_1, T_2, \gamma}(\mathbf{x})_1 &= \gamma \sum_i \frac{C_{i1} g_{T_1}(\mathbf{x}_1) \mathbf{x}_i}{\sum_k C_{ik} g_{T_1}(\mathbf{x}_k)} + (1-\gamma) \frac{\mathbf{d}_1 g_{T_2}(\mathbf{x}_1)}{\sum_k \mathbf{d}_k g_{T_2}(\mathbf{x}_k)} \geq (1-\gamma) \frac{\mathbf{d}_1 g_{T_2}(\mathbf{x}_1)}{\sum_k \mathbf{d}_k g_{T_2}(\mathbf{x}_k)} \\ &\geq \frac{1-\gamma}{1 + \mu e^{-L_{\min}(g_{T_2})(1-2\varepsilon)}}. \end{aligned}$$

Therefore, if $L_{\min}(g_{T_2})(1-2\varepsilon) \geq \ln \frac{\mu(1-\varepsilon)}{\varepsilon-\gamma}$, then $\mathbf{f}_{T_1, T_2, \gamma}(\mathbf{x})_1 \geq 1-\varepsilon$ and $\mathbf{f}_{\infty, T_2, \gamma}(\mathbf{x}) \in \Sigma_\varepsilon$. By Brouwer's Fixed Point Theorem, the continuous map $\mathbf{f}_{T_1, T_2, \gamma}$ has at least one fixed point in Σ_ε . \square

We now consider the case where the damping factor γ approaches 1 and $T_1 = \infty$. Then, the corresponding value of the generalized PageRank converges to an invariant measure of the matrix $\text{diag}(C\mathbf{1})^{-1}C$, independently of the choice of T_2 . In order to prove this, we will need the following classical result, which is a particular case of Corollary 3.1 in [30]. For a matrix $M \in \mathbb{C}^{n \times n}$, the *index*, $\text{ind } M$, is the smallest nonnegative integer k such that $\text{rank } M^{k+1} = \text{rank } M^k$, and its *Drazin inverse*, M^D , is the unique solution X of the equations $M^{k+1}X = M^k$, $XM^kX = X$, $MX = XM$, where $k = \text{ind } M$.

Lemma 4.6 (See [30], Coro. 3.1). *Let $M \in \mathbb{C}^{n \times n}$. If $\text{ind } M = 1$, then $\lim_{\varepsilon \rightarrow 0} \varepsilon(M + \varepsilon I)^{-1} = I - MM^D$.*

Proposition 4.7. *Assume $T_1 = \infty$. For all vector norm $\|\cdot\|$ and all $\varepsilon > 0$, there exists $\gamma_\varepsilon < 1$ such that for every fixed point \mathbf{x} of $\mathbf{f}_{\infty, T_2, \gamma}$, with $\gamma_\varepsilon < \gamma < 1$, there exists an invariant measure \mathbf{u} of $\text{diag}(C\mathbf{1})^{-1}C$ such that $\|\mathbf{x} - \mathbf{u}\| < \varepsilon$.*

Proof. Let $A = \text{diag}(C\mathbf{1})^{-1}C$, and let $M = I - A$. Since A is stochastic, $\text{ind } M = 1$ (see for instance Theorem 8.4.2 in [4]). Therefore, with $\gamma = (1 + \varepsilon)^{-1}$,

$$\lim_{\gamma \rightarrow 1} (1-\gamma)(I - \gamma A)^{-1} = \lim_{\varepsilon \rightarrow 0} \varepsilon(M + \varepsilon I)^{-1} = I - MM^D.$$

Let $\|\cdot\|$ be a vector norm and $\|\cdot\|$ its induced matrix norm, and let $\nu > 0$ such that $\|\mathbf{v}\| \leq \nu$ for all stochastic vector \mathbf{v} . Let $\varepsilon > 0$. There exists $\gamma_\varepsilon < 1$ such that if $\gamma_\varepsilon < \gamma < 1$,

$$\|(1-\gamma)(I - \gamma A)^{-1} - (I - MM^D)\| < \nu^{-1} \varepsilon.$$

Let $\gamma \in]\gamma_\varepsilon, 1[$, and let \mathbf{x} be a fixed point of $\mathbf{f}_{\infty, T_2, \gamma}$, that is $\mathbf{x} = \mathbf{v}(\mathbf{x})(1-\gamma)(I - \gamma A)^{-1}$, where $\mathbf{v}(\mathbf{x})_i = \mathbf{d}_i g_{T_2}(\mathbf{x}_i) / \sum_k \mathbf{d}_k g_{T_2}(\mathbf{x}_k)$ for all i . Then,

$$\|\mathbf{x} - \mathbf{v}(\mathbf{x})(I - MM^D)\| < \varepsilon.$$

But $\mathbf{v}(\mathbf{x})(I - MM^D)$ is an invariant measure of the matrix A . Indeed, $I - MM^D$ is stochastic, and $(I - MM^D)(I - A) = M - MM^D M = M - M^2 M^D = 0$, by definition of the Drazin inverse. \square

5. ESTIMATING THE CRITICAL TEMPERATURE

We call *critical temperature* the larger temperature for which the number of fixed points of \mathbf{u}_T changes. It corresponds to the loss of the uniqueness of the fixed point. In this section, we are interested in estimating the critical temperature for some particular cases. We will study in details the case of $n \times n$ matrices of all ones with the particular weight function $g_T(x) = e^{x/T}$.

We suppose that $g_T(x) = e^{x/T}$ and first consider the particular case where the graph is *complete* with

$$C = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

For this matrix, the point $\mathbf{x} = \frac{1}{n}\mathbf{1}$ is a fixed point of \mathbf{u}_T for all T . We are interested in the existence of other fixed points, depending on the temperature T .

Lemma 5.1. *Assume that C is the $n \times n$ matrix of all ones. The point \mathbf{x} is a fixed point of \mathbf{u}_T if and only if $\mathbf{x} \in \Sigma$ and there exists $\lambda \in \mathbb{R}$ such that $\lambda = \mathbf{x}_i e^{-\mathbf{x}_i/T}$ for every $i = 1, \dots, n$.*

Proof. This follows directly from $\mathbf{x} = \mathbf{f}_T(\mathbf{x})$. \square

Lemma 5.2. *Assume that C is the $n \times n$ matrix of all ones. The point \mathbf{x} is a fixed point of \mathbf{u}_T if and only if $\mathbf{x} \in \Sigma$ and there exists $\mathcal{K} \subseteq \{1, \dots, n\}$, $y \in [0, T]$ and $z \geq T$ such that $ye^{-y/T} = ze^{-z/T}$, $|\mathcal{K}|y + (n - |\mathcal{K}|)z = 1$, $\mathbf{x}_i = y$ for all $i \in \mathcal{K}$ and $\mathbf{x}_i = z$ for all $i \notin \mathcal{K}$.*

Proof. Since the map $x \mapsto xe^{-x/T}$ is increasing for $0 \leq x < T$ and decreasing for $x > T$, there can be at most two values $y \neq z$ such that $ye^{-y/T} = ze^{-z/T} = \lambda$ for a given $\lambda \in \mathbb{R}$. The result hence follows from Lemma 5.1 and $\mathbf{x} \in \Sigma$. \square

Since in our case, $L_{\max}(g_T) = T^{-1}$, we know from Theorem 3.6 that the critical temperature is at most n . Proposition 5.3 shows that this critical temperature is in fact roughly $(\ln n)^{-1}$ when n tends to infinity.

Proposition 5.3. *Assume that C is the $n \times n$ matrix of all ones. If $n > 2$, the map \mathbf{u}_T has a unique fixed point $\mathbf{x} = (\frac{1}{n} \cdots \frac{1}{n})$ if and only if $T > T^*(n)$, where*

$$\frac{1 - \frac{1}{\ln n}}{\ln((\ln n - 1)n + 1)} \leq T^*(n) = \sup_{\alpha > 1} \frac{1 - \frac{1}{\alpha}}{\ln((\alpha - 1)n + 1)} < \frac{1}{\ln(n - 1)},$$

thus $T^(n) \sim \frac{1}{\ln n}$ when n tends to ∞ . If $n = 2$, the map \mathbf{u}_T has a unique fixed point $\mathbf{x} = (\frac{1}{2} \ \frac{1}{2})$ if and only if $T \geq T^*(2) = \frac{1}{2}$.*

Proof. From Lemma 5.2, $\mathbf{x} \in \text{int}(\Sigma)$ is a fixed point of \mathbf{u}_T , with $\mathbf{x} \neq (\frac{1}{n} \cdots \frac{1}{n})$, if and only if there exists $\mathcal{K} \subset \{1, \dots, n\}$, $y, z \in \mathbb{R}$, such that $\mathbf{x}_i = y$ for $i \in \mathcal{K}$, $\mathbf{x}_i = z$ for $i \notin \mathcal{K}$, $0 < k = |\mathcal{K}| < n$, $ky + (n - k)z = 1$, $ye^{-y/T} = ze^{-z/T}$, and $y < z$. Denote $\alpha = \frac{1}{ny}$. Since $y < \frac{1}{n}$, we get necessary that $\alpha > 1$. From $ye^{-y/T} = ze^{-z/T}$, we get $T = T_{\alpha, k}$, where

$$T_{\alpha, k} = \frac{1 - \frac{1}{\alpha}}{(n - k) \ln \left(\frac{(\alpha - 1)n}{n - k} + 1 \right)}.$$

This implies that \mathbf{u}_T has a fixed point $\mathbf{x} \in \text{int}(\Sigma)$, $\mathbf{x} \neq (\frac{1}{n} \cdots \frac{1}{n})$ if and only if $T \in \mathcal{T} = \{T_{\alpha, k}, \alpha > 1, k \in \{1, \dots, n - 1\}\}$. Let

$$T^*(n) = \sup_{\alpha > 1} T_{\alpha, n-1} = \sup_{\alpha > 1} \frac{1 - \frac{1}{\alpha}}{\ln((\alpha - 1)n + 1)}.$$

We shall show that $\mathcal{T} =]0, T^*(n)]$ when $n > 2$ and $\mathcal{T} =]0, T^*(2)[$ when $n = 2$.

First, a study of $T_{\alpha, k}$ as a function of k shows that it is increasing. It is therefore sufficient to show that $\{T_{\alpha, n-1}, \alpha > 1\} =]0, T^*(n)]$ when $n > 2$, and $\{T_{\alpha, 1}, \alpha >$

$1\} =]0, T^*(2)[$ when $n = 2$. Second, a study of $T_{\alpha, n-1}$ as a function of $\alpha > 1$ shows that, when $n > 2$, $T_{\alpha, n-1}$ is increasing, then decreasing, tends to 0 when α goes to infinity, and its maximum is attained for $\alpha = \alpha_n$, where $\alpha_n > \frac{2(n-1)}{n}$. Hence $\mathcal{T} =]0, T^*(n)[$. When $n = 2$, $T_{\alpha, 1}$ is decreasing, tends to 0 when α goes to infinity, and to $\frac{1}{2}$ when α goes to 1. Hence $T^*(2) = \frac{1}{2}$, and $\mathcal{T} =]0, T^*(2)[$.

Moreover, for $n \geq 3$,

$$T^*(n) = T_{\alpha_n, n-1} = \frac{1 - \frac{1}{\alpha_n}}{\ln((\alpha_n - 1)n + 1)} < \frac{1}{\ln((\alpha_n - 1)n + 1)},$$

and since $\alpha_n > \frac{2(n-1)}{n}$, we get $T^*(n) < \frac{1}{\ln(n-1)}$. For the lower bound, we get $T_{\ln n, n-1} \leq T^*(n)$, since $\ln n > 1$. \square

Proposition 5.3 deals with the very special case of a complete graph. In more general circumstances, the exact computation of the critical temperature seems out of range. However, we can obtain numerically a lower bound of the critical temperature, which seems to be an accurate estimate, using the following homotopy-type method. We first choose two random initial vectors on the simplex. Then, we iterate the map \mathbf{f}_T from each of these vectors. For small values of T , this yields with an overwhelming probability two different webranks. Then, we increase the temperature T , and keep iterating the map \mathbf{f}_T on each of these webranks, until the two webranks coincide. This yields a lower bound of the critical temperature. Then, we repeat this procedure, with new random initial vectors, until the lower bound of the critical temperature is not improved any more. Note that the simpler method consisting in keeping T fixed and iterating \mathbf{f}_T from various initial conditions (random vectors or Dirac distributions on a vertex of the simplex) experimentally yields an under estimate of the critical temperature.

Using the previously described homotopy-type method, we computed numerically the critical temperature for two families of graphs. These experiments reveal that the $1/\ln(n)$ asymptotics obtained for the complete graph gives a good general estimate. We first considered the ring graph, with n nodes, in which node i is connected to its two neighbors and to itself. The critical temperature, for $n = 51, 201, 501$ and 1001 is shown by stars in Figure 2. The exact value of the critical temperature of the complete graph with n nodes, $T^*(n)$, is drawn as a continuous curve. We see that the critical temperatures of the ring and complete graphs are essentially proportional. We also computed numerically the critical temperature for a standard model of random directed graph, in which the presence of the different arcs are independent random variables, and for every (i, j) , the probability of presence of the arc (i, j) is given by the same number p . We took $p = 10/n$, so that every node is connected to an average number of 10 nodes. The corresponding critical temperatures are represented by circles. The values of these critical temperatures do not seem to change significantly with the realization of the random graph, hence, each of the values which are represented correspond to a unique realization.

Remark 5.1. *Let us briefly discuss about the case of an arbitrary 2×2 irreducible matrix C with the weight function $g_T = e^{x/T}$. In this case, some elementary calculations give information about the critical temperature [1]. Firstly, the critical temperature for a graph of only two nodes is always less than 1, since it can be shown that \mathbf{u}_T has a unique fixed point if $T \geq 1$. This is the best general upper*

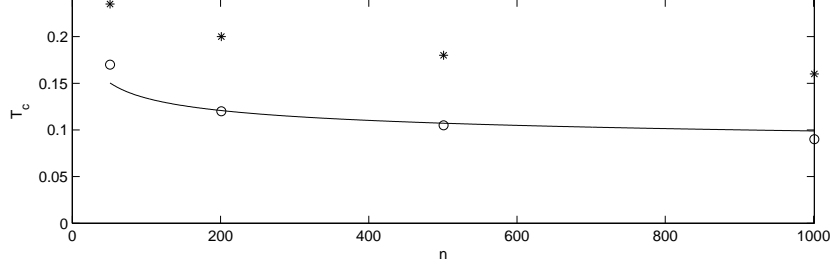


Figure 2: Estimation of the critical temperature T_c as a function of the number of nodes n for the complete graph (continuous curve), the ring graph (stars) and random graphs (circles).

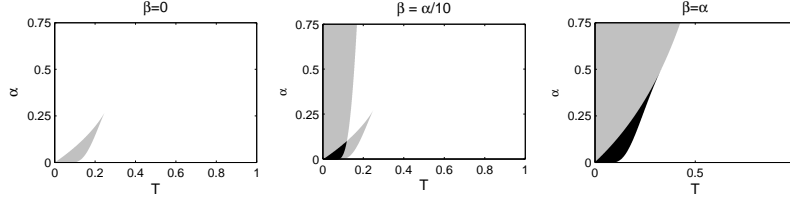


Figure 3: For 2×2 matrices, the map \mathbf{u}_T has 5 fixed points for (T, α) in the black region, 3 fixed points for (T, α) in the grey region, and 1 fixed point otherwise.

bound that can be given for problems of this dimension since for every $T < 1$, we can construct a 2×2 matrix such that \mathbf{u}_T has at least two fixed points. Moreover, one can show that, for every $T > 0$, the map \mathbf{u}_T has at most 5 fixed points and does not have any orbit of period greater than 1. Numerical experiments show that for a 2×2 irreducible matrix, the number of fixed points of the map \mathbf{u}_T can change 0, 1, 2 or even 3 times when decreasing the temperature T .

This can be seen on Figures 3, which were obtained experimentally. Let $\alpha = C_{11}/C_{12}$ and $\beta = C_{22}/C_{21}$. For a specified β , \mathbf{u}_T has 5 fixed point if (T, α) belongs to the black region, 3 fixed points in the grey region and 1 fixed point in the white region.

6. EXPERIMENTS ON A SUBGRAPH OF THE WEB

In this section, we briefly present our experiments of the T -PageRank on a large-scale example. We consider a subgraph of the Web with about 280000 nodes which has been obtained by S. Kamvar from a crawl on the Stanford web¹. We use the variant of our model presented in Section 4, with a transition matrix given by

$$M(\mathbf{x})_{ij} = \gamma \frac{C_{ij} e^{\mathbf{x}_j/T}}{\sum_k C_{ik} e^{\mathbf{x}_k/T}} + (1 - \gamma) \frac{e^{\mathbf{x}_j/T}}{\sum_k e^{\mathbf{x}_k/T}},$$

where we suppose that for each dangling node i (i.e. a node corresponding to a webpage without hyperlink), the i th row of the matrix C is a row of all ones.

¹The adjacency matrix can be found on <http://www.stanford.edu/~sdkamvar/research.html>.

The chosen damping factor is $\gamma = 0.85$. We have computed the T -PageRank from the recurrence (2) for various temperatures T and initial rankings. As expected, when the temperature T is large, the T -PageRank is very close to the classical PageRank, and when T approaches zero, arbitrary close initial rankings can induce totally different T -PageRanks. The critical temperature experimentally seems to be about $T = 0.033$. It has the same order of magnitude as the $T^*(n) = 0.06148$ estimate discussed in Section 5.

As in [38], we represented in a log-log scale the cumulative distribution function of the PageRank, i.e the proportion of pages for which the T -PageRank is larger than a given value, as a function of this value. In Figure (a), we show the successive T -PageRanks obtained for *increasing* temperatures from $T = 0.015$ to the critical temperature $T = 0.033$ by the following variant of the previously described homotopy method: for $T = 0.015$, we iterate the map \mathbf{f}_T , with a Dirac mass on a vertex of the simplex as initial ranking, until a fixed point is reached. Then, for each new value of T , we iterate \mathbf{f}_T until a fixed point is reached, starting from the previous fixed point. For $T \leq 0.032$ the distribution of the T -PageRank is quite different from this of the PageRank and it comes closer suddenly for $T = 0.033$. In Figure (b), we show the successive T -PageRanks obtained by a similar method for *decreasing* temperatures from $T = 0.033$ to 0.009, with the classical PageRank as an initial ranking. The latter procedure may be compared with simulated annealing schemes, in which the temperature is gradually decreased. Until $T = 0.0091$, the distribution of the T -PageRank is quite similar to that of the classical PageRank (see a zoom in Figure (c)). With $T = 0.009$, the T -PageRank moves suddenly away from the PageRank. These figures suggest that the gap between webpages considered as “good” and “bad” is more pronounced with the T -PageRank than with the classical PageRank.

We have also compared the five best nodes for the classical PageRank and for the T -PageRank with decreasing temperatures $T = 0.033$, 0.015 and 0.0091. As we see in Table 1, for $T = 0.033$, the PageRank and T -PageRank give a similar ranking for the top-five. But for smaller temperatures as $T = 0.015$ or $T = 0.0091$, even the two best nodes are exchanged.

$T = 0.033$	$T = 0.015$	$T = 0.0091$
1	2	2
2	1	1
3	3	3
4	6	46
5	7	33

Table 1: The five best nodes of the T -PageRank for several values of T : the numbers refer to the rankings according to the classical PageRank.

Since for this special set of data, the correspondence between the page numbers and the urls is not available, one cannot interpret the discrepancies between the PageRank and the T -PageRank in Table 6.1. In [32], J.-P. Poveda made similar experiments on the larger matrix obtained by S. Kamvar for a crawl of the union of the Stanford and Berkeley Webs², with about 685000 nodes, for which, this

²Also available on <http://www.stanford.edu/~sdkamvar/research.html>.

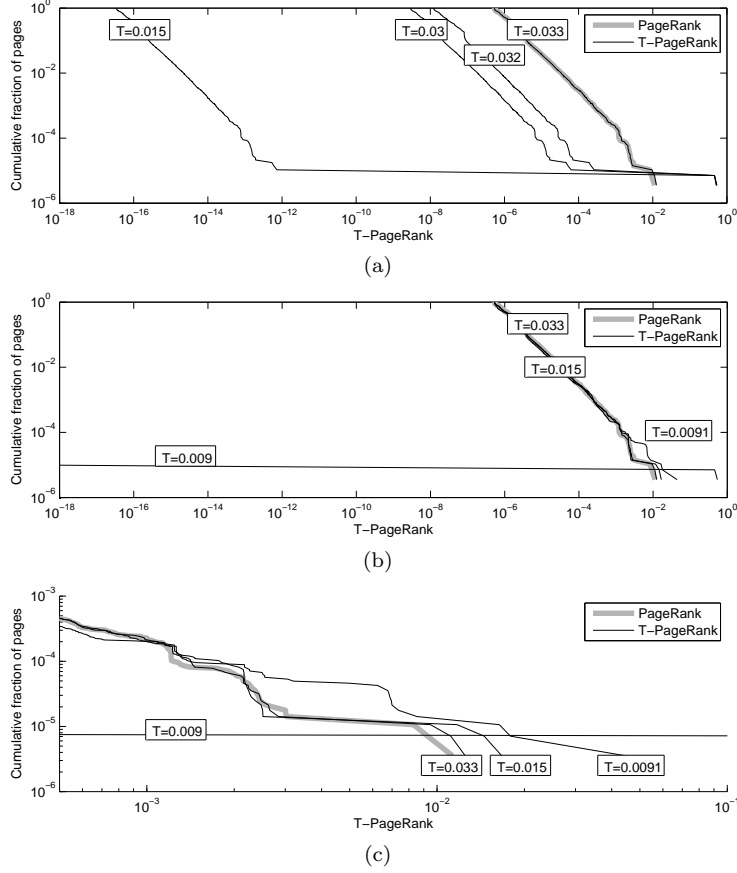


Figure 4: Fraction of pages having a PageRank larger than a particular value. (a) T -PageRanks computed with increasing temperatures from $T = 0.015$ to 0.033 , with a vertex of the simplex as initial ranking. (b) T -PageRanks computed with decreasing temperatures from $T = 0.033$ to 0.009 , with the classical PageRank vector as initial ranking. (c) Zoom of Figure (b).

time, the correspondence between some pages and the main urls is given. These experiments suggest that the T -PageRank obtained by the latter scheme, in which the temperature is gradually decreased, as illustrated in Figure (b), might be of practical interest. A full experimentation, on a real scale web, is beyond the scope of the present paper.

As a concluding remark, we would like to point out that our results might be considered as an argument in favor of the thesis that one should not use PageRank type measures to assess quality. Indeed, the validity of the classical PageRank relies on an ideal view of the web, in which the makers of pages are thought of as experts, creating hyperlinks only to pages they carefully examined, and judged by themselves to be of interest. In the real world, however, the web makers may be influenced by factors like reputation, to which the webrank participates. Within the limits of the model, our results show that pathological phenomena, like getting

non unique and even not meaningful rankings, may occur in a self-referential world in which the websurfers would excessively rely on the web ranking, rather than on their own judgement.

REFERENCES

- [1] M. AKIAN, S. GAUBERT, AND L. NINOVE, *The T-PageRank: a model of self-validating effects of web surfing*, in Positive systems, vol. 341 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 2006, pp. 239–246. Proceedings of the Second Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA'06), Grenoble, France, August 30 – September 1, 2006.
- [2] M. ARTZROUNI, *On the dynamics of the linear process $Y(k) = A(k)Y(k-1)$ with irreducible matrices $A(k)$* , SIAM J. Matrix Anal. Appl., 17 (1996), pp. 822–833.
- [3] M. ARTZROUNI AND O. GAVART, *Nonlinear matrix iterative processes and generalized coefficients of ergodicity*, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1343–1353.
- [4] A. BERMAN AND R. J. PLEMMONS, *Nonnegative matrices in the mathematical sciences*, vol. 9 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994. Revised reprint of the 1979 original.
- [5] M. BOUKLIT AND F. MATHIEU, *BackRank: an alternative for PageRank?*, in WWW05: Special interest tracks and posters of the 14th international conference on World Wide Web, New York, NY, USA, 2005, ACM Press, pp. 1122–1123.
- [6] S. BRIN AND L. PAGE, *The anatomy of a large-scale hypertextual web search engine*, Computer Networks and ISDN Systems, 30 (1998), pp. 107–117. Proceedings of the Seventh International World Wide Web Conference, April 1998.
- [7] R. A. BRUALDI AND H. J. RYSER, *Combinatorial matrix theory*, vol. 39 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1991.
- [8] H. CASWELL, *Matrix Population Models: Construction, Analysis, and Interpretation*, Sinauer, second ed., 2001.
- [9] O. CATONI, *Simulated annealing algorithms and Markov chains with rare transitions*, in Séminaire de Probabilités, XXXIII, vol. 1709 of Lecture Notes in Math., Springer, Berlin, 1999, pp. 69–119.
- [10] J. CONLISK, *Stability and monotonicity for interactive Markov chains*, J. Math. Sociol., 17 (1992), pp. 127–143.
- [11] R. FAGIN, A. R. KARLIN, J. KLEINBERG, P. RAGHAVAN, S. RAJAGOPALAN, R. RUBINFELD, M. SUDAN, AND A. TOMKINS, *Random walks with “back buttons”*, Ann. Appl. Probab., 11 (2001), pp. 810–862.
- [12] S. GAUBERT AND J. GUNAWARDENA, *The Perron-Frobenius theorem for homogeneous, monotone functions*, Trans. Amer. Math. Soc., 356 (2004), pp. 4931–4950 (electronic).
- [13] L. GURVITS, *Stability of discrete linear inclusion*, Linear Algebra Appl., 231 (1995), pp. 47–85.
- [14] D. J. HARTFIEL, *Nonhomogeneous matrix products*, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
- [15] T. H. HAVELIWALA, *Topic-sensitive PageRank*, in WWW02: Proceedings of the 11th international conference on World Wide Web, New York, NY, USA, 2002, ACM Press, pp. 517–526.
- [16] J. A. HOLYST, K. KACPERSKI, AND F. SCHWEITZER, *Social impact models of opinion dynamics*, in Annual Reviews of Computational Physics, Dietrich Stauffer, ed., vol. 9, World Scientific, Singapore, 2001, pp. 253–273.
- [17] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.
- [18] I. C. F. IPSEN AND S. KIRKLAND, *Convergence analysis of a PageRank updating algorithm by Langville and Meyer*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 952–967.
- [19] A. JADBABAIE, J. LIN, AND A. S. MORSE, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Automat. Control, 48 (2003), pp. 988–1001.
- [20] G. JEY AND J. WIDOM, *Scaling personalized web search*, in WWW03: Proceedings of the 12th international conference on World Wide Web, New York, NY, USA, 2003, ACM Press, pp. 271–279.
- [21] S. KAMVAR, T. HAVELIWALA, AND G. GOLUB, *Adaptive methods for the computation of PageRank*, Linear Algebra Appl., 386 (2004), pp. 51–65.

- [22] S. KIRKLAND, *Conditioning of the entries in the stationary vector of a Google-type matrix*, Linear Algebra Appl., 418 (2006), pp. 665–681.
- [23] U. KRAUSE, *Perron’s stability theorem for nonlinear mappings*, J. Math. Econom., 15 (1986), pp. 275–282.
- [24] A. N. LANGVILLE AND C. D. MEYER, *Deeper inside PageRank*, Internet Math., 1 (2004), pp. 335–380.
- [25] ———, *Google’s PageRank and beyond: the science of search engine rankings*, Princeton University Press, Princeton, NJ, 2006.
- [26] ———, *Updating Markov chains with an eye on Google’s PageRank*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 968–987.
- [27] J. LORENZ, *A stabilization theorem for dynamics of continuous opinions*, Phys. A, 355 (2005), pp. 217–223.
- [28] ———, *Consensus strikes back in the hegselmann-krause model of continuous opinion dynamics under bounded confidence*, Journal of Artificial Societies and Social Simulation, 9 (2006).
- [29] V. METZ, *Nonlinear Perron-Frobenius theory in finite dimensions*, Nonlinear Anal., 62 (2005), pp. 225–244.
- [30] C. D. MEYER, JR., *Limits and the index of a square matrix*, SIAM J. Appl. Math., 26 (1974), pp. 469–478.
- [31] R. D. NUSSBAUM, *Hilbert’s projective metric and iterated nonlinear maps*, Mem. Amer. Math. Soc., 75 (1988), pp. iv+137.
- [32] J. P. POVEDA, *Calcul du T-PageRank d’un graphe du Web*, PPL internship report, École Nationale Supérieure des Techniques Avancées, Août 2007.
- [33] M. RICHARDSON AND P. DOMINGOS, *The intelligent surfer: Probabilistic combination of link and content information in PageRank*, in Advances in Neural Information Processing Systems 14 (NIPS2001), MIT Press, 2001, pp. 1441–1448.
- [34] E. SENETA, *Nonnegative matrices and Markov chains*, Springer Series in Statistics, Springer-Verlag, New York, second ed., 1981.
- [35] S. SERRA-CAPIZZANO, *Jordan canonical form of the Google matrix: a potential contribution to the PageRank computation*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 305–312.
- [36] M. SYDOW, *Random surfer with back step*, in WWW Alt.04: Proceedings of the 13th international World Wide Web conference on Alternate track papers & posters, New York, NY, USA, 2004, ACM Press, pp. 352–353.
- [37] W. T. TUTTE, *Graph theory*, vol. 21 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2001. Reprint of the 1984 original.
- [38] Y. V. VOLKOVICH, N. LITVAK, AND D. DONATO, *Determining factors behind the PageRank log-log plot*, Memorandum 1823, University of Twente, Enschede, February 2007.
- [39] J. WOLFOWITZ, *Products of indecomposable, aperiodic, stochastic matrices*, Proc. Amer. Math. Soc., 14 (1963), pp. 733–737.

INRIA, DOMAINE DE VOLUCEAU, BP105, 78153 LE CHESNAY CÉDEX, FRANCE
E-mail address, Marianne Akian: **Marianne.Akian@inria.fr**

INRIA, DOMAINE DE VOLUCEAU, BP105, 78153 LE CHESNAY CÉDEX, FRANCE
E-mail address, Stéphane Gaubert: **Stephane.Gaubert@inria.fr**

DEPARTMENT OF MATHEMATICAL ENGINEERING, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, 1348 LOUVAIN-LA-NEUVE, BELGIUM
E-mail address, Laure Ninove: **Laure.Ninove@uclouvain.be**